DETERMINANTS

DEFINITION

When an algebraic or numerical expression is expressed in a square form containing some rows and columns, this square form is named as a determinant of that expression. For example when expression $a_1b_2 - a_2 b_1$ is expressed in the form

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$

then it is called a determinant of order 2, Clearly a determinant of order 2 contains 2 rows and 2 columns. Similarly

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$ is a determinant of order 3.

Obviously in every determinant, the number of rows and columns are equal and this number is called the order of that determinant.

REPRESENTATION OF A DETERMINANT

Generally we use Δ or $|A|$ symbols to express a determinant and a determinant of order 3 is represented by

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

It should be noted that the (i, j)th element (i.e., the element of the i^{th} row and j^{th} column) of the determinant has been expressed by a_{ij} , $i = 1 , 2 , 3 ; j = 1,2,3$. The elements for which $i = j$ are called diagonal elements and the diagonal containing them is called principal diagonal or simply diagonal of the determinant. For the above determinant a_{11} , a_{22} , a_{33} are diagonal elements.

A determinant is called a triangular determinant if its every element above or below the diagonal is zero. For example

$$\begin{vmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{vmatrix}$$

is a triangular determinant. In particular when all the elements except diagonal elements are zero, then it is called a diagonal determinant. For example

$$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

is a diagonal determinant.

We generally use R_1, R_2, R_3, ... to denote first, second, third .... row and C_1, C_2 , C_3 .... to denote first, second, third ..... column of a determinant.

VALUE OF A DETERMINANT

The expression which has been expressed in a determinant form is called the value of that determinant.

To find the value of a third order determinant

Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

be a third order determinant. To find its value we expand it by any row or column as the sum of three determinants of order 2. If we expand it by first row then

$$\Delta = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

MINOR AND COFACTOR OF AN ELEMENT

MINOR OF AN ELEMENT

Minor of an element of the determinants is obtain by leaving the row and column containing that element and retaining rest of elements.

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then minor of a_{11} is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$. Similarly $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Using this concept the value of Determinant can be

$\Delta = a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13}$

or $\Delta = -a_{21} M_{21} + a_{22} M_{22} - a_{23} M_{23}$

or $\Delta = a_{31} M_{31} - a_{32} M_{32} + a_{33} M_{33}$

COFACTOR OF AN ELEMENT

The cofactor of an element a_{ij} is denoted by C_{ij} and is equal to $(-1)^{i+j} M_{ij}$ where M_{ij} is a minor of element a_{ij}

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

then $C_{11} = (-1)^{1+1} M_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Note :- (i) The sum of products of the element of any row with their corresponding cofactor is equal to the value of determinant i.e. $\Delta = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$

(ii) The sum of the product of element of any row with corresponding cofactor of another row is equal to zero i.e. $a_{11} C_{21} + a_{12} C_{22} + a_{13} C_{23} = 0$

(iii) If order of a determinant (Δ) is 'n' then the value of the determinant formed by replacing every element by its cofactor is Δ^{n-1}

PROPERTIES OF DETERMINANTS

If the elements of a determinant are complicated expressions or numbers, then it is very difficult to find its value by expansion method. In such cases we reduce the determinant into a simple one using the following properties.

P–1 The value of a determinant is unchanged if its rows and columns are interchanged. For example

$$\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} = \begin{vmatrix} a & p & u \\ b & q & v \\ c & r & w \end{vmatrix}$$

P–2 The interchange of any two consecutive rows or columns will simply change the sign of the value of the determinant. For example

$$\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} = -\begin{vmatrix} b & a & c \\ q & p & r \\ v & u & w \end{vmatrix} = -\begin{vmatrix} p & q & r \\ a & b & c \\ u & v & w \end{vmatrix}$$

P–3 If any two rows or columns of a determinant are identical then its value is zero. For example

$$\begin{vmatrix} a & b & c \\ a & b & c \\ u & v & w \end{vmatrix} = 0 = \begin{vmatrix} a & a & b \\ p & p & q \\ u & u & v \end{vmatrix}$$

P–4 If each element of a row or column of a determinant be multiplied by a number, then its value is also multiplied by that number. For example

$$\begin{vmatrix} ka & kb & kc \\ p & q & r \\ u & v & w \end{vmatrix} = k\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} = \begin{vmatrix} ka & b & c \\ kp & q & r \\ ku & v & w \end{vmatrix}$$

P–5 If each entry in a row or column of a determinant is the sum of two numbers, then the determinant can be written as the sum of two determinants. For example

$$\begin{vmatrix} a+\alpha & b+\beta & c+\gamma \\ p & q & r \\ u & v & w \end{vmatrix} = \begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} + \begin{vmatrix} \alpha & \beta & \gamma \\ p & q & r \\ u & v & w \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a+\alpha & b & c \\ p+\beta & q & r \\ u+\gamma & v & w \end{vmatrix} = \begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} + \begin{vmatrix} \alpha & b & c \\ \beta & q & r \\ \gamma & v & w \end{vmatrix}$$

P–6 The value of a determinant does not change if the elements of a row (column) are added to or subtracted from the corresponding elements of another row (column). For example

$$\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} = \begin{vmatrix} a+\alpha b+\beta c & b & c \\ p+\alpha q+\beta r & q & r \\ u+\alpha v+\beta w & v & w \end{vmatrix}$$

P–7 If Δ = f (x) and f(a) = 0, then (x –a) is a factor of Δ. For example in the determinant

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \quad \text{if we replace a by b then } \Delta = \begin{vmatrix} 1 & 1 & 1 \\ b & b & c \\ b^2 & b^2 & c^2 \end{vmatrix} = 0$$

$\Rightarrow$ (a– b) is a factor of Δ.

P–8 If each entry in any row (or column) of a determinant is zero, then the value of determinant is equal to zero.

MULTIPLICATION OF TWO DETERMINANTS

Multipication of two second order determinants is defined as follows

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \ell_1 & m_1 \\ \ell_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1\ell_1 + b_1\ell_2 & a_1m_1 + b_1m_2 \\ a_2\ell_1 + b_2\ell_2 & a_2m_1 + b_2m_2 \end{vmatrix}$$

Multiplication of two third order determinants is defined as follows

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{vmatrix} = \begin{vmatrix} a_1\ell_1 + b_1\ell_2 + c_1\ell_3 & a_1m_1 + b_1m_2 + c_1m_3 & a_1n_1 + b_1n_2 + c_1n_3 \\ a_2\ell_1 + b_2\ell_2 + c_2\ell_3 & a_2m_1 + b_2m_2 + c_2m_3 & a_2n_1 + b_2n_2 + c_3n_3 \\ a_3\ell_1 + b_3\ell_2 + c_3\ell_3 & a_3m_1 + b_3m_2 + c_3m_3 & a_3n_1 + b_3n_2 + c_3n_3 \end{vmatrix}$$

Note : In above case the order of Determinant is same, if the order is different then for their multiplication first of all they should be expressed in the same order

SYMMETRIC & SKEW SYMMETRIC DETERMINANT

Symmetric determinant

A determinant is called symmetric Determinant if for its every element.

$a_{ij} = a_{ji} \ \forall \ i, j$

Skew Symmetric determinant

A determinant is called skew Symmetric determinant if for its every element

$a_{ij} = -a_{ji} \ \forall \ i, j$;

Note : (i) Every diagonal element of a skew symmetric determinant is always zero

(ii) The value of a skew symmetric determinant of even order is always a perfect square and that of odd order is always zero.

i.e. (order = 2) i.e. (order = 3)

APPLICATIONS OF DETERMINANT

CRAMMER'S RULE :

Let the system of equations be

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

and $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$ $\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, $\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

then (i) If $\Delta \neq 0$, then given system of equations is consistent i.e. has unique solution and its solution is

$$\boxed{x = \frac{\Delta_1}{\Delta}, \ y = \frac{\Delta_2}{\Delta}, \ z = \frac{\Delta_3}{\Delta}}$$

This is known as Crammer's rule

(ii) If $\Delta = 0$ and atleast one of $\Delta_1, \Delta_2, \Delta_3$ is not zero, then the system of equations is inconsistent i.e. it has no solution.

(iii) If $\Delta = 0$ and $\Delta_1 = 0 = \Delta_2 = \Delta_3$, then the system has infinite solutions.

(iv) If $\Delta = 0$ and $d_1 = 0 = d_2 = d_3$, then the system of equations has infinite solutions (non-zero solution) i.e. non-trivial solutions

(v) If $\Delta \neq 0$ and $d_1 = 0 = d_2 = d_3$, then the system of equations has a unique solution $x = 0$, $y = 0$, $z = 0$ i.e., zero solution or trivial solution.

DIFFERENTIATION OF A DETERMINANT :

Let $\Delta(x) = \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix}$,

then $\Delta'(x) = \begin{vmatrix} f_1'(x) & g'_1(x) & h'_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f'_2(x) & g'_2(x) & h'_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f'_3(x) & g'_3(x) & h'_3(x) \end{vmatrix}$

INTEGRATION OF DETERMINATION

Let $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ p & q & r \\ l & m & n \end{vmatrix}$, where p, q, r, l, m and n are constants.

Then $\int_a^b \Delta(x)dx = \begin{vmatrix} \int_a^b f(x)dx & \int_a^b g(x)dx & \int_a^b h(x)dx \\ p & q & r \\ l & m & n \end{vmatrix}$

USE OF SUMMATION

If $f(r) = \begin{vmatrix} r & r^2 & r^3 \\ p & q & t \\ 1 & 2 & 3 \end{vmatrix}$, where p, q, t are constants, then $\sum_{r=1}^{n} f(r) = \begin{vmatrix} \sum_{r=1}^{n} r & \sum_{r=1}^{n} r^2 & \sum_{r=1}^{n} r^3 \\ p & q & t \\ 1 & 2 & 3 \end{vmatrix}$